BOUNDARY BEHAVIOR OF THE KOBAYASHI-ROYDEN METRIC IN SMOOTH PSEUDOCONVEX DOMAINS

Peter Pflug, Włodzimierz Zwonek

Abstract. We show some lower estimates for the Kobayashi-Royden metric on a class of smooth bounded pseudoconvex domains.

1. Introduction and main results.

In this short note we discuss the problem of the boundary behavior of the Kobayashi-Royden metric (mainly) in the normal direction in smooth bounded pseudoconvex domains. We show two main results. One of the results states in particular that the Kobayashi-Royden metric in the normal direction in some class of smooth bounded pseudoconvex domains is estimated from below by the expression like $1/d_D^{7/8}(z)$ ($d_D(z)$ denotes the distance of z from the boundary of D) which improves a recent result of S. Fu (see [Fu 2009]) where the Author obtained the lower estimate with the exponent $5/6$. On the other hand we show that a careful study of a recent example of J. E. Fornaess and L. Lee (see [Lee 2009]) shows that the optimal exponent in the lower estimate of the Kobayashi-Royden metric in the normal direction is smaller than one (for C^k-smoothness, $k<\infty$) and we also show some obstacles for the rate of the increase in the C^∞-case.

Recall that the Kobayashi-Royden metric has a localization property (see e.g. [Roy 1971], [Gra 1975]); therefore, we lose no generality concentrating on domains defined globally. Recall also that one of reasons to study the boundary behavior of the Kobayashi-Royden metric is the problem of deciding whether any bounded smooth pseudoconvex domain is Kobayashi complete (see e.g. [Jar-Pfl 1993]). The hope was that the Kobayashi-Royden metric in the normal direction explodes near the boundary as $1/d_D(z)$; it was one of the ideas to show that smooth bounded pseudoconvex domains are Kobayashi complete. However, after many years of uncertainty a recent example of Fornaess-Lee ([For-Lee 2009]) showed that such a lower bound is not valid. More precisely, the example is the following.

Theorem 1 (see [For-Lee 2009]). *For any given increasing sequence $(a_\nu)_\nu$, $a_\nu\to\infty$, of positive numbers there is a bounded smooth pseudoconvex domain $D\subset\mathbb{C}^3$ and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu\to 0$ such that*

$$\kappa_D(P_{\delta_\nu};n)\leq 1/(a_\nu\delta_\nu),$$

Both authors were supported by a DFG grant No 436 POL 113/106/0-2 (July 2009/September 2009) and by the the Research Grant No. N N201 361436 of the Polish Ministry of Science and Higher Education.

2000 Mathematics Subject Classification. Primary: 32F45

keywords: Kobayashi-Royden metric, pseudoconvex domain

where P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

In the other direction S. Fu showed that smooth bounded pseudoconvex domains have the following lower estimate (our formulation is weaker than the one in the original paper).

Theorem 2 (see [Fu 2009]). *Let D be a bounded C^3-smooth pseudoconvex domain given by the formula $D = \{r < 0\}$, where r is a C^3-smooth defining function (meaning that its Levi form is semipositive definite on the complex tangent space of any boundary point). Then there is a constant $c > 0$ such that*

$$\kappa_D(z;X) \geq c\frac{|\langle \partial r(z), X\rangle|}{|r(z)|^{2/3}}, \quad z \in D,\ X \in \mathbb{C}^n.$$

Moreover, it follows from the paper of S. Fu that if we make some additional assumption on the vector X and points z (for instance X is the unit outward normal vector to some boundary point P and z lies on the line passing through P in the direction X) then in the above estimate we may replace the exponent $2/3$ with $5/6$. We shall see in Theorem 4 that in many cases the exponent $5/6$ may be replaced by $7/8$.

S. Fu also conjectured that in the class of smooth domains the lower estimate of the Kobayashi-Royden metric as in Theorem 2 may be taken of the form $1/d_D^{1-\varepsilon}(z)$ with $\varepsilon > 0$ arbitrarily small. Note that the example of Fornaess-Lee shows that the exponent cannot be taken to be equal to one (equivalently ε cannot be equal to 0).

However, the careful study of the example of Fornaess-Lee shows that in the case of C^k-smooth domains an estimate as conjectured by S. Fu does not hold. We have namely the following result.

Theorem 3. *(1) For any positive integer k there are a C^k-smooth bounded pseudoconvex domain D in $\mathbb{C}^3$, a positive number ε, and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that*

$$\kappa_D(P_{\delta_\nu}; n) \leq 1/\delta_\nu^{1-\varepsilon},$$

where P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

(2) For any $\alpha > 0$ there are a C^∞-smooth bounded pseudoconvex domain D in $\mathbb{C}^3$ and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that

$$\kappa_D(P_{\delta_\nu}; n) \leq \frac{1}{\delta_\nu(-\log\delta_\nu)^\alpha},$$

where P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

Note that the above theorem shows that even in the C^∞ case the proof of the Kobayashi completeness of the smooth bounded pseudoconvex domain cannot go along the following lines: We prove that the Kobayashi-Royden metric (in the normal direction) behaves like a 'regular' integrable function of $d_D(z)$. Therefore, in case that all smooth bounded pseudoconvex domains are all Kobayashi complete the proof would require a more subtle reasoning.

We can, however, say also something in the positive direction. Namely, we may slightly improve the estimate given in Theorem 2. Unfortunately, the better estimate holds for smooth domains defined as sublevel sets of smooth plurisubharmonic defining function - recall that not all smooth bounded pseudoconvex domains are locally sublevel sets of smooth plurisubharmonic functions (see [For 1979], [Beh 1985]).

Theorem 4. *Let $D = \{r < 0\}$ be a bounded domain in $\mathbb{C}^n$ where $r : U \mapsto \mathbb{R}$ is a C^4-smooth plurisubharmonic defining function for D. Then there is a constant $C > 0$ such that*

$$\kappa_D(z;X) \geq \frac{C|\langle n(z),X\rangle|}{d_D^{7/8}(z)}$$

as z tends to ∂D and the vectors X are taken so that $||X|| = o(1/d_D(z))|\langle n(z),X\rangle|$, where $n(z)$ denotes the unit outward normal vector to ∂D at the point of ∂D of the smallest distance from z.

Before we start the proofs recall the definition of the Kobayashi-Royden (pseudo)metric of a domain $D \subset \mathbb{C}^n$ (for basic properties of the Kobayashi-Royden metric see [Jar-Pfl 1993]).

$$\kappa_D(z;X) := \inf\{\alpha > 0 : \text{ there is an } f \in \mathcal{O}(\mathbb{D}, D) \text{ with } f(0) = z,\ \alpha f'(0) = X\},$$

$z \in D,\ X \in \mathbb{C}^n$.

2. Proofs. We start with some preliminary considerations.

Let $D = \{r < 0\}$ be a domain in $\mathbb{C}^n$, where $r : U \mapsto \mathbb{R}$ is a C^{k+1}-smooth plurisubharmonic defining function with $r(0) = 0$. Then (up to a linear isomorphism) the Taylor expansion at 0 of order k of r is of the following form

$$r(z) = \operatorname{Re} z_n + \sum_{j=2}^{k} Q_j(z) + R_k(z),$$

where $Q_j(z) = \sum_{|\alpha|+|\beta|=j} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta$ (note that then $a^j_{\alpha,\beta} = \bar{a}^j_{\beta,\alpha}$). We may also write $Q_j(z) = \tilde{Q}_j(z) + \hat{Q}_j(z)$, where

$$\tilde{Q}_j(z) = \sum_{|\alpha|+|\beta|=j,|\alpha|,|\beta|>0} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta, \quad \hat{Q}_j(z) = 2\operatorname{Re} \sum_{|\alpha|=j} a^j_{\alpha,\alpha} z^\alpha =: 2\operatorname{Re} H_j(z).$$

It follows from Taylor's formula that $\mathcal{L}R_k(z)|_{S^{2n-1}} = O(||z||^{k-1})$, where $\mathcal{L}\tilde{r}(z)(X)$ is the Levi form of $\tilde{r}$ at the point z in direction of X; S^{2n-1} means the $2n-1$-dimensional sphere. Consequently, $\mathcal{L}R_k(z)(z) = O(||z||^{k+1})$.

An easy calculation gives the following formula

$$\mathcal{L}Q_j(z)(z) = \sum_{\nu=0}^{j} \nu(j-\nu) \sum_{|\alpha|=\nu,|\beta|=j-\nu} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta =$$

$$\sum_{\nu=1}^{j-1} \nu(j-\nu) \sum_{|\alpha|=\nu,|\beta|=j-\nu} a^j_{\alpha,\beta} z^\alpha \bar{z}^\beta.$$

In particular, $\mathcal{L}Q_2(z)(z) = \tilde{Q}_2(z)$, $\mathcal{L}Q_3(z)(z) = 2\tilde{Q}_3(z)$.

In the sequel we shall denote by C_j different constants that depend only on the domain D.

Proof of Theorem 4. We leave the notation as above and we make use of the above considerations.

First we prove the desired estimate but with the exponent equal to $3/4$ (instead of $7/8$).

The assumptions of the theorem imply that for $z \in U$ we have the following estimate

$$\tilde{Q}_2(z) + 2\tilde{Q}_3(z) + C_1||z||^4 \geq 0,$$

which together with the property

$$\min\{\tilde{Q}_2(z)+\tilde{Q}_3(z), \tilde{Q}_2(-z)+\tilde{Q}_3(-z)\} \geq \min\{\tilde{Q}_2(z)+2\tilde{Q}_3(z), \tilde{Q}_2(-z)+2\tilde{Q}_3(-z)\}$$

gives for z close to 0 the inequality

$$r(z) \geq \operatorname{Re} z_n + Q_2(z) + Q_3(z) - C_2||z||^4 \geq \operatorname{Re} z_n + 2\operatorname{Re}(H_2(z) + H_3(z)) - C_3||z||^4.$$

Therefore, shrinking U if necessary, we have the following inclusion

$$D \subset \{\operatorname{Re} z_n + 2\operatorname{Re}(H_2(z) + H_3(z)) - C_3||z||^4 < 0\}.$$

Now for $\delta > 0$ small enough and $X \in \mathbb{C}^n$, $X \neq 0$, take $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such that $\varphi(0) = (0, \dots, 0, -\delta)$, $\kappa\varphi'(0) = X$ where $\kappa > 0$.

Note that $||\varphi(\lambda) - \varphi(0)|| \leq C_4|\lambda|$. For $r \in (0,1)$ define $\psi_r(\lambda) := \varphi(r\lambda)$. Then $||\psi_r(\lambda)|| \leq \delta + C_4 r$, $\lambda \in \mathbb{D}$.

Define $\Psi(z) := z_n + 2H_2(z) + 2H_3(z)$. Put $\varphi_r := \Psi \circ \psi_r$. Then $\varphi_r(\mathbb{D}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < C_5(\delta + r)^4\} =: S_{r,\delta}$ for δ small enough.

Therefore,

$$\frac{r}{\kappa}|X_n(1+O(\delta)) + \sum_{j=1}^{n-1} O(\delta)X_j| = \frac{r}{\kappa}\,|\Psi'(0,\dots,0,-\delta)X| = |\varphi_r'(0)| \leq C_6(\delta+(\delta+r)^4),$$

where the last inequality follows easily from the formula for the Kobayashi-Royden metric for $S_{r,\delta}$.

Substitute $r = \delta^{1/4}$ for δ small enough. Then we get the following lower estimate

$$\kappa \geq C_7\delta^{1/4}\frac{|X_n|(1+\alpha(\delta))}{\delta + (\delta + \delta^{1/4})^4},$$

where $\alpha(\delta) \to 0$ as $\delta \to 0$ (here we use the fact that we choose X such that $||X|| = o(1/\delta)|X_n|$). Consequently, we get the following lower estimate

$$\kappa_D((0,\dots,0,-\delta); X) \geq C_6|X_n|/\delta^{3/4}$$

as δ tends 0 and the vectors are taken so that $o(1/\delta)|X_n| \geq ||X||$.

Our aim is to show that we may replace the exponent 3/4 with 7/8. Keeping in mind the above estimate and leaving the same notation as above we have the following inequality

$$||\varphi(\lambda)-\varphi(0)-\frac{\lambda}{\kappa}X||\leq C_7|\lambda|^2.$$

Proceeding as before we get for $|\lambda|\leq r$, δ small enough

$$||\varphi(\lambda)||\leq\delta+\frac{rC_8}{\kappa}||X||+C_7r^2$$

or $||\psi_r(\lambda)||\leq\delta+\frac{rC_8}{\kappa}||X||+C_7r^2$, $\lambda\in\mathbb{D}$.

Since without loss of generality we may assume that $||X||$ is bounded from above (or even equal to one) proceeding exactly as before we get the following inequality

$$\kappa\geq\frac{C_9r|X_n|(1+\alpha(\delta))}{\delta+(\delta+\frac{r}{\kappa}+r^2)^4}.$$

Since we already know that $\kappa\geq C_{10}/(\delta^{3/4})$ (at this place we need the first part of the proof), putting $r=\delta^{1/8}$ we get the following estimate

$$\kappa_D((0,\dots,0,-\delta);X)\geq\frac{C_{11}|X_n|}{\delta^{7/8}}$$

as δ tends to 0 and X satisfies the inequality $||X||\leq o(1/\delta)|X_n|$, which finishes the proof of the theorem. $\square$

Remark 5. Consider r to be defined near 0 as follows $r(z):=\operatorname{Re}z_2+p(z_1)$ where

$$p(z_1):=2m^2z_1^{m+l}\bar{z}_1^{m-l}+4(m^2-l^2)|z_1|^{2m}+2m^2z_1^{m-l}\bar{z}_1^{m+l}=\\|z_1|^{2m}(\operatorname{Re}4m^2e^{i2l\theta}+4(m^2-l^2)),$$

$m/2\leq l<m$ (here $z_1=e^{i\theta}|z_1|$). Since p is a subharmonic function such that for some values of θ the last factor in the formula is negative (the example is taken from [Las 1988]) we see that we cannot hope to repeat the reasoning from the proof of Theorem 4 for general k - even the case $k=2m=4$ ($m=2$, $l=1$) encounters an obstacle. In other words the above method of the proof does not give a better lower estimate. Nevertheless, we think that the lower estimate in the normal direction of the Kobayashi-Royden metric near the boundary of a C^{k+2}-smooth pseudoconvex domain may be of the form $1/d_D^{1-1/(2k)}(z)$, which would mean that in the case of infinitely smooth bounded pseudoconvex domain the estimate with the exponent arbitrarily close to 1 (as suggested by S. Fu) may hold.

Remark 6. In the proof of Theorem 4 one needs twice the same reasoning. However, instead of repeating it twice one may use a result of S. Fu (to get the lower estimate of κ of the form $1/\delta^{2/3}$ - in fact it is sufficient to have the estimate of the form $1/\delta^{1/8}$). However, at the present form the proof is more self contained. Therefore, the authors decided to leave it in the present form.

Proof of Theorem 3. As mentioned earlier the domain which satisfies the properties claimed in the theorem was constructed in [For-Lee 2009]. Therefore, we recall the

construction from there (keeping the notation from there, too). To get the proof of the theorem we have to add some estimates (mostly for derivatives) of the defined functions and also at some places, for simplicity of calculations, we make some special choice of some sequences.

First let us make some comments. Note that the procedure works not only for $r_{n+1}=\frac{r_n^2}{a_n}$ but also under the assumption that $r_{n+1}\leq\frac{r_n^2}{a_n}$, also the choice of A_k may be done with the equality replaced by the inequality. Consequently, the series that we shall choose can be replaced by any subseries.

Below we shall write some inequalities for norms. Such an inequality: $l_n\leq m_n$ means that $\limsup\frac{l_n}{m_n}<\infty$. The meaning of the equality is analogous (inequalities in both directions hold). The norms of functions are meant to be the supremum norms of functions (on some sets).

At first stage we repeat the definition of a sequence of subharmonic functions which is then adopted to the construction of a sequence of subharmonic functions of two variables defining a three-dimensional example. As mentioned earlier the construction follows entirely from [For-Lee 2009].

At first we assume the existence of sequences $(a_n)_n$, $(r_n)_n$ such that the sequence $(a_n)_n$ is increasing to infinity and $r_{n+1}\leq\frac{r_n^2}{a_n}$. We shall fix the sequences later.

We define

$$u_n(z):=1/8-\operatorname{Re}z+\frac{\log|z|}{4\log a_n},\ z\in\mathbb{C}.$$

Then we

$$R_n(z):=\begin{cases}\max\{u_n(z),0\}, & \operatorname{Re}z\leq b_n\\ u_n(z), & \operatorname{Re}z>b_n\end{cases},$$

where $0<b_n\leq 1$ is the smallest positive number such that

$$1/8-b_n+\frac{\log b_n}{4\log a_n}=0.$$

At first we are interested in the norm of R_n on a closed disc of radius Ma_n/r_n (for some fixed $M>1$). It is estimated from above by a_n/r_n.

We define

$$\tilde{R}_n(z):=\int_{\mathbb{C}}R_n(z-\varepsilon_n w)\chi(w)d\mu(w)$$

for some $0<\varepsilon_n<r_n/2$, where $\mu=dxdy/m$. Here $m=\int_{\mathbb{C}}\chi(z)dxdy$ and $\chi:\mathbb{C}\mapsto[0,\infty)$ is a non-constant C^∞ radial function such that $0\leq\chi\leq 1$ and $\chi(z)=0$ for $|z|\geq 1$.

Then $||\tilde{R}_n^{(k)}||_{B(0,M)}\leq(1/r_n)^k\cdot a_n/r_n$. Then we put $\rho_n(z):=\tilde{R}_n(\frac{a_nz}{r_n})$.

We therefore have

$$||\rho_n^{(k)}||_{B(0,M)}\leq\left(\frac{a_n}{r_n^2}\right)^k\frac{a_n}{r_n}$$

.

At this place we fix the sequences. We put $r_n=1/a_n$. We also want to have $r_{n+1}:=\frac{r_n^2}{a_n}=r_n^3$. In other words we may choose $r_n:=r_1^{3^n}$. Now fix for a while $\varepsilon\in(0,1/3)$. And put $a_n=\delta_n^{-\varepsilon}$. So finally, the choice of the numbers is the following $a_n=a^{\varepsilon 3^n}$, where $a>1$ is fixed, $r_n=(1/a)^{\varepsilon 3^n}$, $\delta_n=(1/a)^{3^n}$. But the

construction needs also additional number $A_n = 1/2 + a_n/r_n + \log(1/r_n)/4\log a_n$ such that

$$\delta_n \leq \frac{\delta_{n-1}}{A_n \frac{1}{2^n}}.$$

With our choice of numbers we get that $A_n = a^{2\varepsilon 3^n}$ (in the asymptotic sense). The construction needs also that $\delta_n \leq \frac{\delta_{n-1}}{A_n 2^n}$. Since the inequality must hold asymptotically (it follows from the reasoning) it is sufficient to see that for large n

$$a^{-3^n} \leq \frac{1}{a^{3^{n-1}+2\varepsilon 3^n} 2^n},$$

which holds for ε as above.

At this place the one-dimensional function ρ is defined as

$$\rho(z) := \sum_{n=1}^{\infty} \delta_n \rho_n(z),$$

which defines a C^k-smooth function under the assumption

$$\sum_n \delta_n ||\rho_n^{(k)}||_{B(0,M)} < \infty.$$

In other words this gives the condition $\sum_n \delta_n \frac{a_n^{k+1}}{r_n^{2k+1}} < \infty$. But the last series is $\sum a^{((3k+2)\varepsilon-1)3^n}$, which is finite when $\varepsilon < 1/(3k+2)$.

Now we move to the construction of the proper function $\tilde{\rho}$.

We define $V := \{(s,t) \in \mathbb{C}^2 : s^2 - t^3 = 0\}$. We want to have $\tilde{\rho}_n(s,t) = \rho_n(s/t) = \rho_n(\zeta)$ if $(s,t) = (\zeta^3, \zeta^2) \in V$.

Let $\tilde{r}_n := r_{n+1}^3$ and put $B_n := B(0, \tilde{r}_n) \subset \mathbb{C}^2$, we also put $B_n' := B(0, 3/4\tilde{r}_n)$. Then one may choose a small neighborhood U_n of V such that the projection $\pi : U_n \mapsto V$ is well-defined on $U_n \backslash B_n'$ (the formula is the following $\pi(s,t) := (s, s^{2/3})$ with a properly chosen branch of the power). We put $U_n := \{p \in \mathbb{C}^2 : ||p - \pi(p)|| < d_n^2\}$, where one may choose $d_n = r_{n+1}^3 = r_n^9$ (asymptotically in the above mentioned sense).

Then $||\pi^{(k)}||_{U_n \backslash B_n'} = r_{n+1}^{-k} = \frac{1}{r_n^{3k}}$.

We define $\tilde{\rho}_n := \rho_n \circ \pi$ on $U_n \setminus B_n$ and we may extend $\tilde{\rho}_n$ to a C^∞-smooth on $B_n \cup U_n$ letting it be equal 0 on B_n.

Now we note the next estimate

$$||\tilde{\rho}_n^{(k)}||_{(U_n \backslash B_n) \cap B(0,M)} \leq \frac{1}{r_n^{6k+2}}.$$

Let $\chi : \mathbb{R} \mapsto [0,1]$ be a C^∞-smooth function, equal to 1 on $[0,1/2]$ and equal to 0 on $[1,\infty)$.

Then we define another smooth extension of $\tilde{\rho}_n$ on $\mathbb{C}^2$ by the formula

$$p_n(z) := \begin{cases} 0, & z \in B_n, \\ \tilde{\rho}_n(z), & z \in U_n \setminus B_n, ||z - \pi(z)|| \leq \frac{d_n^2}{2} \\ \tilde{\rho}_n(z)\chi\left(\frac{||z-\pi(z)||^2}{d_n^2}\right), & z \in U_n \setminus B_n, \frac{d_n^2}{2} \leq ||z-\pi(z)||^2 \leq d_n^2, \\ 0, & z \notin U_n \cup B_n \end{cases}$$

Then one may verify that $||p_n^{(k)}||_{B(0,M)} \leq \frac{1}{r_n^{9k+2}}$. Now we take $C_n \geq 0$ such that $\mathcal{L}p_n(z)(X) \geq -C_n||X||^2$. It follows that we may take (asymptotically)

$$C_n = \frac{1}{r_n^{20}}$$

– use the estimate for the norm of p_n''.

Put

$$A_n := \{z \in U_n \setminus B_n : \frac{d_n^2}{2} \leq ||z - \pi(z)||^2 \leq d_n^2\}$$

and

$$q(s,t) := e^{||(s,t)||^2}|s^2 - t^3|^2,\ (s,t) \in \mathbb{C}^2.$$

Note that $\mathcal{L}q(s,t)(X) \geq |s^2 - t^3|^2||X||^2$. Therefore, if we take (asymptotically) $c_n = d_n^2 = r_n^{18}$, then

$$\mathcal{L}q(z)(X) \geq c_n||X||^2.$$

Put $K_n = \frac{1}{r_n^{38}}$ (asymptotically). Then $-C_n + K_n c_n \geq 0$. Consequently, $\tilde{r}_n = p_n + K_n q$ is plurisubharmonic on $\mathbb{C}^2$. Certainly, $||\tilde{r}_n^{(k)}||_{B(0,M)} \leq \max\{\frac{1}{r_n^{9k+2}}, \frac{1}{r_n^{38}}\} =: \frac{1}{r_n^{m_k}}$ (note that $m_k = 38$, $k = 1, \dots, 4$ and $m_k = 9k + 2$, $k \geq 4$). Now the condition on C^k-smoothness of the example from [For-Lee 2009] follows from the C^k-smoothness of

$$\tilde{\rho} := \sum_n \delta_n \tilde{r}_n,$$

which is satisfied if

$$(*) \qquad \infty > \sum_n \delta_n \frac{1}{r_n^{m_k}} = \sum_n a^{m_k\varepsilon - 1}.$$

The last inequality completes the proof with arbitrary $\varepsilon \in (0, 1/m_k)$.

To complete the construction recall that Fornaess and Lee defined the domain as follows

$$D := \{(s,t,w) \in \mathbb{C}^3 : \operatorname{Re} w + \tilde{\rho}(s,t) < 0\} \cap B(0,2).$$

Let us now move to the second part of the theorem.

We leave all the relations between the numbers a_n, δ_n, r_n with one exception. Namely, put $a_n = (-\log\delta_n)^\alpha$, $\alpha > 0$. Explicitly we have $\delta_n = (1/a)^{3^n}$, $r_n = 1/a_n$. Then the convergence of the final sequence $(*)$ (with just introduced δ_n and r_n) is easily satisfied. And although the relation $r_{n+1} \leq \frac{r_n^2}{a_n}$ is not satisfied now, it is easy to see that instead of taking the whole sequence while defining the function $\tilde{\rho}$ we may also choose an arbitrary subsequence which easily guarantees that the desired inequality is satisfied. One may easily prove that choosing these relations the number A_n satisfies the desired inequality as well.

Above considerations lead us to the following relation being sufficient for the construction of a C^∞-smooth domain with the boundary behaviour of the Kobayashi-Royden metric in the normal direction equal to $1/(\delta_n(-\log\delta_n)^\alpha)$:

$$\sum \delta_n a_n^k < \infty \text{ for any positive integer } k.$$

The last condition is, as one may verify, satisfied. $\square$

References

[Beh 1985] M. Behrens, *Plurisubharmonic defining functions of weakly pseudoconvex domains in $\mathbb{C}^2$*, Math. Ann. **270** (1985), 285–296.

[For 1979] J. E. Fornaess, *Plurisubharmonic defining functions*, Pacific J. Math. **80** (1979), 381–388.

[For-Lee 2009] J. E. Fornaess, L. Lee, *Asymptotic behavior of Kobayashi metric in the normal direction*, Math. Z. **261** (2009), 399–408.

[Fu 2009] S. Fu, *The Kobayashi metric in the normal direction and the mapping problem*, Complex Var. Elliptic Equ. **54** (2009), 303–316.

[Gra 1975] I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc. **207** (1975), 219–240.

[Jar-Pfl 1993] M. Jarnicki, P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, 1993.

[Las 1988] G. Laszlo, *Peak functions on finite type domains in $\mathbb{C}^2$, PhD Thesis, Eötvös Loránd University, Budapest* (1988).

[Roy 1971] H. Royden, *Remarks on the Kobayashi metric*, Several Complex Variables II, Lecture Notes in Math. **185** (1971), Springer, 125–137.

Carl von Ossietzky Universität Oldenburg, Institut für Mathematik, Postfach 2503, D-26111 Oldenburg, Germany

E-mail address: `pflug@ mathematik.uni-oldenburg.de`

Instytut Matematyki, Uniwersytet Jagielloński, Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: `Wlodzimierz.Zwonek@ im.uj.edu.pl`